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SIMILARITY LAWS FOR LIFTING BODIES OF MINIMUM DRAG AT HYPERSONIC SPEEDS

BY

ANGELO MIELE

RICE UNIVERSITY

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SUMMARY

The problem of determining the longitudinal and transversal contours minimizing the drag of a flat-top body in hypersonic flow is considered under the hypotheses that the pressure distribution is Newtonian and the skin-friction coefficient is constant. It is also assumed that the body is slender and homothetic and that certain arbitrarily prescribed values are assigned to -- at most -- the lift, the pitching moment, the planform area, the frontal area, the wetted area, the volume, the length, and the thickness. Two similarity laws are determined.

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(**) Professor of Astronautics and Director of the Aero-Astronautics Group, Department of Mechanical and Aerospace Engineering and Materials Science, Rice University, Houston, Texas.

The Similarity Law for Longitudinal Contours permits one to determine the optimum longitudinal contour of a body of arbitrary transversal contour from the known optimum longitudinal contour of a reference body (a body of semicircular cross section); the aerodynamic and geometric quantities of the latter must be replaced by appropriate proportional quantities of the former, with the proportionality constants depending only on the prescribed transversal contour.

The Similarity Law for Transversal Contours permits one to determine the optimum transversal contour of a body of arbitrary longitudinal contour from the known optimum transversal contour of a reference body (a conical body); the aerodynamic and geometric quantities of the latter must be replaced by appropriate proportional quantities of the former, with the proportionality constants depending only on the prescribed longitudinal contour.

1. INTRODUCTION

In a previous report (Ref. 1), the basic theory of slender, lifting bodies in the hypersonic regime was formulated under the following hypotheses: (a) a plane of symmetry exists between the left-hand and right-hand sides of the body; (b) no plane of symmetry exists between the upper and lower parts; however, the intersection of these parts is a curve contained in a plane perpendicular to the plane of symmetry, called the reference plane; (c) the base plane is perpendicular to both the plane of symmetry and the reference plane; (d) the body is slender in the longitudinal sense, that is, the square of the slope of any meridian contour is small with respect to one; (e) the body is homothetic, in the sense that each cross section is geometrically similar to the base section and has the same orientation; (f) the free-stream velocity is perpendicular to the base plane and, therefore, is parallel to the line of intersection of the plane of symmetry and the reference plane; (g) the pressure coefficient is twice the cosine squared of the angle formed by the free-stream velocity and the normal to each surface element; (h) the skin-friction coefficient is constant; and (i) the contribution of the tangential forces to the lift is negligible with respect to the contribution of the normal forces. Direct methods were employed in Ref. 1. Specifically, the class of flat-top bodies whose longitudinal contours are power laws and whose transversal contours are semielliptical or triangular was considered, and the thickness ratio, the power law exponent, and the elongation ratio of the cross section were determined so as to maximize the lift-to-drag ratio.

It should be noted that the above direct methods supply results which are valid for only particular longitudinal and transversal contours. Therefore, it is important to reformulate the minimal problem by using the indirect methods of the calculus of variations, that is, by eliminating any previous restriction on the class of bodies being investigated. Thus, two complementary variational problems arise: (1) to determine the optimum longitudinal contour for an arbitrarily prescribed transversal contour; and (2) to determine the optimum transversal contour for an arbitrarily prescribed longitudinal contour. In each case, the quantity to be minimized is the drag, and constraints may be imposed on aerodynamic quantities (lift and pitching moment) and geometric quantities (planform area, frontal area, wetted area, volume, length, and thickness).

Since the number of possible variational problems is practically without limit, economy of thought leads one to pose the following questions: (1) Is there any similarity law which permits one to determine the optimum longitudinal contour of a body of arbitrary transversal contour from the known optimum longitudinal contour of a reference body? and (2) Is there any similarity law which permits one to determine the optimum transversal contour of a body of arbitrary longitudinal contour from the known optimum transversal contour of a reference body? It is the purpose of this paper to show that these similarity laws do exist. By means of them, the results obtained in Ref. 2 for nonlifting bodies are extended to the more important case of lifting bodies.

2. AERODYNAMICS QUANTITIES

In order to relate the drag, the lift, and the pitching moment of a body to its geometry, we define two coordinate systems (Fig. 1): a Cartesian coordinate system Oxyz and a cylindrical coordinate system $Oxr\theta$. For the Cartesian coordinate system, the origin O is the apex of the body; the x-axis is the intersection of the plane of symmetry and the reference plane, positive toward the base; the z-axis is contained in the plane of symmetry, perpendicular to the x-axis, and positive downward; and the y-axis is such that the xyz-system is right-handed. For the cylindrical coordinate system, r is the distance of any point from the x-axis, and θ measures the angular position of this point with respect to the xy-plane.

We refer to the class of flat-top bodies (Fig. 2) whose lower surface is described by the relationship

$$r = r(x, \theta) \tag{1}$$

Thus, in the light of hypotheses (a) through (d) and (f) through (i) of the introduction, the drag D, the lift L, and the pitching moment M per unit free-stream dynamic pressure q are given by (Ref. 1)

$$
\begin{aligned}
D/q &= 2\int_0^{\ell}\int_0^{\pi/2}\left[2r^3 r_x^3/(r^2 + r_\theta^2) + C_f\sqrt{r^2 + r_\theta^2}\right] dx\, d\theta + 2C_f\int_0^{\ell} r(x,0)\, dx \\
L/q &= 4\int_0^{\ell}\int_0^{\pi/2}\left[r^2 r_x^2/(r^2 + r_\theta^2)\right](r\sin\theta - r_\theta\cos\theta)\, dx\, d\theta \qquad (2) \\
M/q &= 4\int_0^{\ell}\int_0^{\pi/2}\left[x r^2 r_x^2/(r^2 + r_\theta^2)\right](r\sin\theta - r_\theta\cos\theta)\, dx\, d\theta
\end{aligned}
$$

where ℓ is the length of the body and C_f the skin-friction coefficient.

2.1. Homothetic Body. Next, we employ hypothesis (e) and focus our attention on the class of bodies such that any transversal contour is geometrically similar to the base contour and has the same orientation. The geometry of these homothetic bodies is given by

$$r = \ell(\tau/\mu)A(\xi)B(\theta) \qquad (3)$$

where

$$\tau = \frac{r(\ell, \pi/2)}{\ell} \qquad (4)$$

denotes the thickness ratio evaluated in the meridian plane $\theta = \pi/2$ and

$$\mu = \frac{r(\ell, \pi/2)}{r(\ell, 0)} \qquad (5)$$

denotes the elongation ratio evaluated in the transversal plane $x = \ell$. Also, $\xi = x/\ell$ denotes a nondimensional abscissa, $A(\xi)$ a function describing the longitudinal contour, and $B(\theta)$ a function describing the transversal contour such that

$$A(0) = 0\,, \qquad A(1) = 1$$
$$B(0) = 1\,, \qquad B(\pi/2) = \mu \tag{6}$$

In the light of Eq. (3), the drag, the lift, and the pitching moment of a homothetic body can be written as (Ref. 1)

$$D/q = \ell^2 \tau \left(\tau^3 I_1 J_1 + C_f\, I_2 J_2 \right)$$
$$L/q = \ell^2 \tau^3 I_3 J_3 \tag{7}$$
$$M/q = \ell^3 \tau^3 I_4 J_4$$

where $I_1, \ldots, I_4$ denote the following integrals depending on the longitudinal contour:

$$I_1 = \int_0^\ell A\dot{A}^3\, d\xi$$
$$I_2 = \int_0^\ell A\, d\xi$$
$$I_3 = \int_0^\ell A\dot{A}^2\, d\xi \tag{8}$$
$$I_4 = \int_0^\ell \xi A\dot{A}^2\, d\xi$$

and J_1 , ..., J_4 denote the following integrals depending on the transversal

contour:

$$
\begin{aligned}
J_1 &= (4/\mu^4)\int_0^{\pi/2}\left[B^6/(B^2+\dot{B}^2)\right]d\theta \\
J_2 &= (2/u)\left[1+\int_0^{\pi/2}\sqrt{B^2+\dot{B}^2}\,d\theta\right] \\
J_3 &= (4/u^3)\int_0^{\pi/2}\left[B^4/(B^2+\dot{B}^2)\right](B\sin\theta-\dot{B}\cos\theta)\,d\theta \\
J_4 &= (4/u^3)\int_0^{\pi/2}\left[B^4/(B^2+\dot{B}^2)\right](B\sin\theta-\dot{B}\cos\theta)\,d\theta
\end{aligned}
\qquad (9)
$$

3. GEOMETRIC QUANTITIES

For the class of flat-top bodies whose lower surface is described by Eq. (1), the planform area S, the frontal area S_f, the wetted area S_w, and the volume V are given by

$$
\begin{aligned}
S &= 2\int_0^{\ell} r(x,0)\,dx \\
S_f &= \int_0^{\pi/2} r^2(\ell,\theta)\,d\theta \\
S_w &= 2\int_0^{\ell}\int_0^{\pi/2} \sqrt{r^2 + r_\theta^2}\,dx\,d\theta + 2\int_0^{\ell} r(x,0)\,dx \\
V &= \int_0^{\ell}\int_0^{\pi/2} r^2\,dx\,d\theta
\end{aligned}
\tag{10}
$$

In particular, if the body is homothetic, Eqs. (10) become

$$
\begin{aligned}
S &= \ell^2 \tau I_5 J_5 \\
S_f &= \ell^2 \tau^2 I_6 J_6 \\
S_w &= \ell^2 \tau\, I_7 J_7 \\
V &= \ell^3 \tau^2 I_8 J_8
\end{aligned}
\tag{11}
$$

where

$$
\begin{aligned}
I_5 &= \int_0^{\ell} A\, d\xi \\
I_6 &= 1 \\
I_7 &= \int_0^{\ell} A\, d\xi \\
I_8 &= \int_0^{\ell} A^2\, d\xi
\end{aligned}
\tag{12}
$$

and

$$
\begin{aligned}
J_5 &= 2/u \\
J_6 &= (1/u^2) \int_0^{\pi/2} B^2\, d\theta \\
J_7 &= (2/u) \left[1 + \int_0^{\pi/2} \sqrt{B^2 + \dot{B}^2}\, d\theta \right] \\
J_8 &= (1/u^2) \int_0^{\pi/2} B^2\, d\theta
\end{aligned}
\tag{13}
$$

Other possible constraints may have the form

$$
\text{Const} = \ell \qquad \text{and/or} \qquad \text{Const} = \ell\tau \tag{14}
$$

meaning the length and/or the thickness are given.

4. SIMILARITY LAW FOR OPTIMUM LONGITUDINAL CONTOURS

Now, suppose that the transversal contour $B(\theta)$ is given, meaning that the quantities $J_1, \ldots, J_8$ are known a priori. Assume that the skin-friction coefficient is given and that certain arbitrarily prescribed values are assigned to -- at most -- the lift, the pitching moment, the planform area, the frontal area, the wetted area, the volume, the length, and the thickness. Observing that the right-hand sides of Eqs. (7), (11), and (14) depend on the parameters ℓ, τ and the longitudinal contour $A(\xi)$ through the integrals $I_1, \ldots, I_8$, we formulate the following variational problem: "In the class of parameters ℓ, τ and functions $A(\xi)$ which satisfy the conditions (6-1), (7-2), (7-3), (11), and (14), find that particular set which minimizes the total drag (7-1)".

Rather than solving this problem as stated, we introduce the following modified aerodynamic and geometric quantities:

$$\tilde{D} = (2\pi/J_1)D, \qquad \tilde{C}_f = \left[2\pi J_2/(2+\pi)J_1\right] C_f$$
$$\tilde{L} = (4/J_3)L, \qquad \tilde{M} = (4/J_4)M$$
$$\tilde{S} = (2/J_5)S, \qquad \tilde{S}_f = (\pi/2J_6)S_f \tag{15}$$
$$\tilde{S}_w = \left[(2+\pi)/J_7\right] S_w, \qquad \tilde{V} = (\pi/2J_8)V$$

and rewrite Eqs. (7) and (11) in the form

$$\tilde{D}/2\pi q = \ell^2 \tau \left[\tau^3 I_1 + (2+\pi)\tilde{C}_f I_2/2\pi\right]$$
$$\tilde{L}/4q = \ell^2 \tau^3 I_3$$
$$\tilde{M}/4q = \ell^3 \tau^3 I_4$$
$$\tilde{S}/2 = \ell^2 \tau \; I_5 \tag{16}$$
$$2\tilde{S}_f/\pi = \ell^2 \tau^2 I_6$$
$$\tilde{S}_w/(2+\pi) = \ell^2 \tau I_7$$
$$2\tilde{V}/\pi = \ell^3 \tau^2 I_8$$

We observe that, since the transversal contour is given, the modified skin-friction coefficient $\tilde{C}_f$, lift $\tilde{L}$, pitching moment $\tilde{M}$, planform area $\tilde{S}$, frontal area $\tilde{S}_f$, wetted area $\tilde{S}_w$, and volume $\tilde{V}$ are known a priori. Furthermore, the modified drag $\widetilde{D}$ is proportional to the actual drag. This being the case, the previous minimal problem is equivalent to that of finding, in the class of parameters ℓ, τ and functions $A(\xi)$ which satisfy the conditions (6-1), (14), and (16-2) through (16-7), that particular set which minimizes the functional (16-1).

We note that, for a flat-top body of semicircular cross section, that is, for

$$B = 1, \qquad 0 \leq \theta \leq \pi/2 \tag{17}$$

the expressions (9) and (13) become

$$\begin{aligned} J_1 &= 2\pi, & J_2 &= 2+\pi, & J_3 &= 4, & J_4 &= 4 \\ J_5 &= 2, & J_6 &= \pi/2, & J_7 &= 2+\pi, & J_8 &= \pi/2 \end{aligned} \tag{18}$$

with the implication that

$$
\begin{aligned}
D/2\pi q &= \ell^2 \tau \left[\tau^3 I_1 + (2+\pi) C_f I_2 / 2\pi \right] \\
L/4q &= \ell^2 \tau^3 I_3 \\
M/4q &= \ell^3 \tau^3 I_4 \\
S/2 &= \ell^2 \tau \; I_5 \\
2S_f/\pi &= \ell^2 \tau^2 I_6 \\
S_w/(2+\pi) &= \ell^2 \tau I_7 \\
2V/\pi &= \ell^3 \tau^2 I_8
\end{aligned}
\qquad (19)
$$

Since Eqs. (16) and (19) are formally identical, the following Similarity Law for Longitudinal Contours can be stated: "The parameters ℓ, τ and the function $A(\xi)$ which optimize the longitudinal contour of a body of arbitrary cross section are identical with those optimizing the longitudinal contour of a body of semi-circular cross section providing the aerodynamic and geometric quantities of the latter are replaced by the modified aerodynamic and geometric quantities (15) of the former".

5. SIMILARITY LAW FOR OPTIMUM TRANSVERSAL CONTOURS

Now, suppose that the longitudinal contour $A(\xi)$ is given, meaning that the quantities $I_1, \ldots\ldots, I_8$ are known a priori. Assume that the skin-friction coefficient is given and that certain arbitrarily prescribed values are assigned to -- at most -- the lift, the pitching moment, the planform area, the frontal area, the wetted area, the volume, the length, and the thickness. Observing that the right-hand sides of Eqs. (7), (11), and (14) depend on the parameters ℓ, τ and the transversal contour $B(\theta)$ through the integrals $J_1, \ldots, J_8$, we formulate the following variational problem: "In the class of parameters ℓ, τ and functions $B(\theta)$ which satisfy the conditions (6-2), (7-2), (7-3), (11), and (14), find that particular function which minimizes the total drag (7-1)".

Rather than solving this problem as stated, we introduce the following modified aerodynamic and geometric quantities:

$$\begin{aligned}
\widetilde{D} &= (1/2\, I_1)\, D\,, & \widetilde{C}_f &= (I_2/I_1)\, C_f \\
\widetilde{L} &= (1/2\, I_3)\, L\,, & \widetilde{M} &= (1/3\, I_4)\, M \\
\widetilde{S} &= (1/2\, I_5)\, S\,, & \widetilde{S}_f &= (1/I_6)\, S_f \\
\widetilde{S}_w &= (1/2\, I_7)\, S_w\,, & \widetilde{V} &= (1/3\, I_8)\, V
\end{aligned} \tag{20}$$

and rewrite Eqs. (7) and (11) in the form

$$\begin{aligned}
2\,\widetilde{D}/q &= \ell^2 \tau\, (\tau^3 J_1 + \widetilde{C}_f J_2) \\
2\,\widetilde{L}/q &= \ell^2 \tau^3 J_3 \\
3\,\widetilde{M}/q &= \ell^3 \tau^3 J_4 \\
2\,\widetilde{S} &= \ell^2 \tau\, J_5 \\
\widetilde{S}_f &= \ell^2 \tau^2 J_6 \\
2\,\widetilde{S}_w &= \ell^2 \tau\, J_7 \\
3\,\widetilde{V} &= \ell^3 \tau^2 J_8
\end{aligned} \tag{21}$$

We observe that, since the longitudinal contour is given, the modified skin-friction coefficient $\widetilde{C}_f$, lift $\widetilde{L}$, pitching moment $\widetilde{M}$, planform area $\widetilde{S}$, frontal area $\widetilde{S}_f$, wetted area $\widetilde{S}_w$, and volume $\widetilde{V}$ are known a priori. Furthermore, the modified drag $\widetilde{D}$

is proportional to the actual drag. This being the case, the previous minimal problem is equivalent to that of finding, in the class of parameters ℓ, τ and functions $B(\theta)$ which satisfy the conditions (6-2), (14), and (21-2) through (21-7), that particular set which minimizes the functional (21-1).

We note that, for a conical body, that is, for

$$A = \xi, \qquad 0 \leq \xi \leq 1 \tag{22}$$

the expressions (8) and (12) become

$$\begin{aligned} I_1 &= 1/2, & I_2 &= 1/2, & I_3 &= 1/2, & I_4 &= 1/3 \\ I_5 &= 1/2, & I_6 &= 1, & I_7 &= 1/2, & I_8 &= 1/3 \end{aligned} \tag{23}$$

with the implication that

$$
\begin{aligned}
2\,D/q &= \ell^2\tau\,(\tau^3 J_1 + C_f\,J_2) \\
2\,L/q &= \ell^2\tau^3 J_3 \\
3\,M/q &= \ell^3\tau^3 J_4 \\
2\,S &= \ell^2\tau\,J_5 \\
S_f &= \ell^2\tau^2 J_6 \\
2\,S_w &= \ell^2\tau\,J_7 \\
3\,V &= \ell^3\tau^2 J_8
\end{aligned}
\tag{24}
$$

Since Eqs. (21) and (24) are formally identical, the following Similarity Law for Transversal Contours can be stated: "The parameters ℓ, τ and the function $B(\theta)$ which optimize the transversal contour of a body of arbitrary longitudinal contour are identical with those optimizing the transversal contour of a conical body providing the aerodynamic and geometric quantities of the latter are replaced by the modified aerodynamic and geometric quantities (20) of the former".

6. DISCUSSION AND CONCLUSIONS

In the previous sections, the minimum drag problem is considered for the class of flat-top, slender, homothetic bodies flying at hypersonic speeds under the assumptions that the pressure distribution is Newtonian and the skin-friction coefficient is constant. It is also assumed that certain arbitrarily prescribed values are assigned to -- at most -- the lift, the pitching moment, the planform area, the frontal area, the wetted area, the volume, the length, and the thickness. Under these hypotheses, two similarity laws are obtained.

The Similarity Law for Longitudinal Contours permits one to determine the optimum longitudinal contour of a body of arbitrary cross section from the known optimum longitudinal contour of a reference body (a body of semicircular cross section). Conversely, the Similarity Law for Transversal Contours permits one to determine the optimum transversal contour of a body of arbitrary longitudinal contour from the known optimum transversal contour of a reference body (a conical body). Finally, the simultaneous use of these similarity laws leads to the idea that -- by combining the results valid for flat-top bodies of semicircular cross section with those valid for conical bodies -- truly three-dimensional, slender, homothetic, lifting bodies of minimum drag can be determined.

REFERENCES

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2. MIELE, A., Similarity Laws for Optimum Hypersonic Bodies, Rice University, Aero-Astronautics Report No. 4, 1965.

LIST OF CAPTIONS

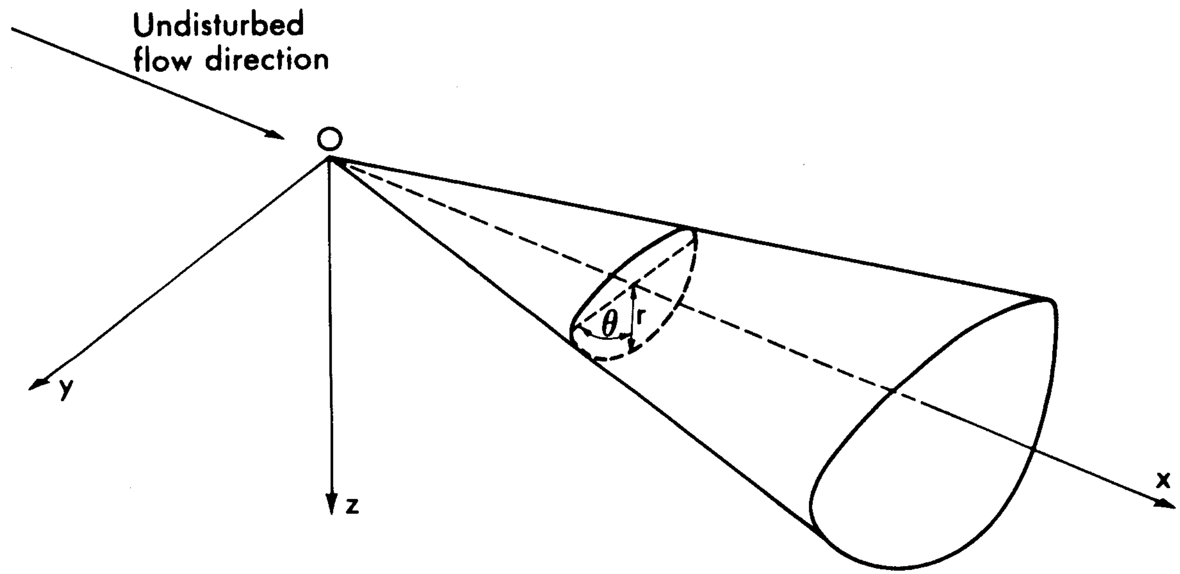

Fig. 1 Coordinate system.

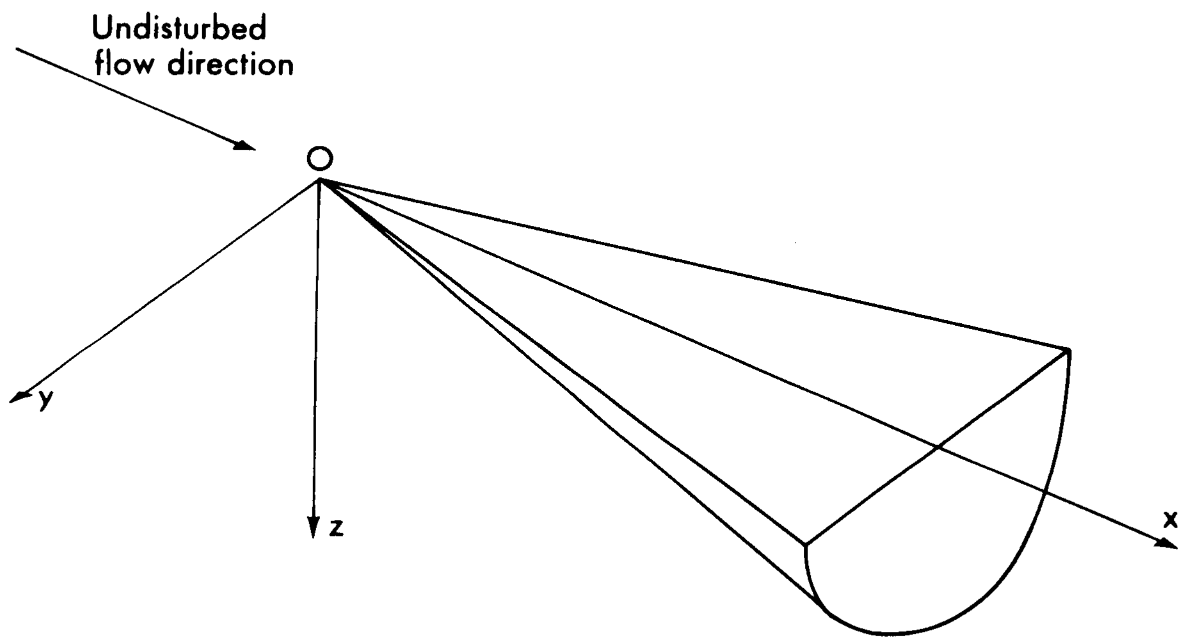

Fig. 2 Flat-top body.